A mean-square bound for the lattice discrepancy of bodies of rotation with flat points on the boundary

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Dedicated to Professor Edmund Hlawka on his 90th birthday

1. Introduction. Let $\mathcal{B}$ denote a compact convex body in s-dimensional Euclidean space, $s \geq 2$, which contains the origin as an inner point and whose boundary $\partial\mathcal{B}$ is sufficiently smooth. The central question of the classic *lattice point theory of large domains* consists of estimating the lattice point discrepancy of a linearly dilated copy $t\,\mathcal{B}$, i.e.,

$$P_{\mathcal{B}}(t) := \#\,(t\,\mathcal{B} \cap \mathbb{Z}^s) - \mathrm{vol}(\mathcal{B})t^s\,, \tag{1.1}$$

where t is a large real parameter. For enlightening accounts on this topic, the reader is referred to E. Krätzel's monographs [14] and [15], to a recent survey article by A. Ivić, E. Krätzel, M. Kühleitner, and W.G. Nowak [10], and to M. Huxley's book [7] where he exposed his breakthrough in the planar case ("Discrete Hardy-Littlewood method").

For the case that $\partial\mathcal{B}$ is of bounded nonzero Gaussian curvature throughout, the usual and plausible conjecture is that

$$P_{\mathcal{B}}(t) \ll t^{\Theta_s+\varepsilon} \tag{1.2}$$

for every $\varepsilon > 0$, where $\Theta_2 := \frac{1}{2}$, $\Theta_s := s-2$ for $s \geq 3$. It is well-known that for every dimension, Θ_s is the minimal possible value with this property, and that (1.2) is actually true for spheres of dimension $s \geq 4$, even with $\varepsilon = 0$ if $s \geq 5$: See, e.g., E. Krätzel [15], p. 227. Quite recently, V. Bentkus and F. Götze [1] and F. Götze [3] established (1.2) for arbitrary ellipsoids of dimension $s \geq 5$.

However, for $s = 2$ and 3, and for general bodies of higher dimensions, the proof or disproof of (1.2) remains an open problem. The sharpest known estimates are due to W. Müller [24]. Researchers subsequently dealt with the task to verify (1.2) "on average", i.e., to show that

$$\int\limits_0^T (P_{\mathcal{B}}(t))^2\,\mathrm{d}t \ll T^{2\Theta_s+1+\varepsilon}\,. \tag{1.3}$$

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In fact, (1.3) was established for planar domains by D.G. Kendall [13] and the author [25], ultimately in the form of an asymptotics [26]. For dimensions $s \geq 4$, (1.3) was proved by W. Müller [23] who however had to leave open the case $s = 3$. This gap was filled by A. Iosevich, E. Sawyer, and A. Seeger [8] who showed that

$$\int_0^T (P_{\mathcal{B}}(t))^2 \, \mathrm{d}t \ll \begin{cases} T^{2\Theta_s+1} & \text{for } s \geq 4, \\ T^3 (\log T)^2 & \text{for } s = 3. \end{cases} \tag{1.4}$$

The very last estimate comes rather close to the asymptotic formula known for the three-dimensional sphere $\mathcal{B}_0$, namely

$$\int_0^T (P_{\mathcal{B}_0}(t))^2 \, \mathrm{d}t = C\, T \, \log T + O\left(T \, (\log T)^{1/2}\right) .$$

Cf. V. Jarnik [11], and also Y.-K. Lau [22] who improved the error term to $O(T)$.

2. Recent developments and statement of present result. The topic of this note will combine two recent trends in lattice point theory: On the one hand, increased interest arose in $\mathbb{R}^3$-bodies *of rotation* (with respect to one of the coordinate axes), denoted by $\mathcal{R}$ in what follows. For the case of nonzero curvature, F. Chamizo [2] obtained the upper bound

$$P_{\mathcal{R}}(t) \ll t^{11/8+\varepsilon} , \tag{2.1}$$

while papers by M. Kühleitner [20] and M. Kühleitner and W.G. Nowak [21] provided Ω-results. A recent article of E. Krätzel and W.G. Nowak [19] gives a version of (2.1) with numerical constants, for the special case of an ellipsoid.

On the other hand, a number of papers investigated the influence of boundary points with curvature zero on the lattice discrepancy. While Krätzel's monograph [15] provides an enlightening survey on the planar case (which is comparatively well understood), results for dimension 3 and higher can be found in the works of K. Haberland [4], E. Krätzel [16], [17], [18], and M. Peter [27]. These are all (pointwise) O-estimates, partially providing a precise evaluation of the contribution of an isolated flat point on $\partial\mathcal{R}$ to $P_{\mathcal{R}}(t)$, with a remainder of smaller order.

In the present paper we shall take up both of these matters, under the aspect of a mean-square estimate in the sense of (1.3). We will consider a $\mathbb{R}^3$-body of rotation $\mathcal{R}$ (with respect to one of the coordinate axes), with smooth boundary $\partial\mathcal{R}$ of nonzero Gaussian curvature κ throughout, except for the points of intersection of $\partial\mathcal{R}$ with the axis of rotation, where κ may vanish. It will turn out that the contribution of these flat points to the lattice point discrepancy can be evaluated quite accurately, leaving a remainder term which is in mean-square "as small as it should be", in the sense of formula (1.3).

We remark parenthetically that if κ would vanish anywhere else on $\partial\mathcal{B}$, it would do so on a whole circle. This will presumably have a more dramatic effect on the lattice discrepancy. It seems much more difficult to obtain a sharp result in this general case.

Precise formulation of present assumptions. Let $\rho : [0,\pi] \to \mathbb{R}_{>0}$ be a function of class C^4, with $\rho'(0) = \rho'(\pi) = 0$ and [(1)]

$$\rho\,\rho'' - 2\rho'^2 - \rho^2 \neq 0 \tag{2.2}$$

throughout $]0,\pi[$. Suppose that ρ is analytic in π and 0. At these two values, the left-hand side of (2.2) may vanish, of orders (exactly) $N_1, N_2 \geq 0$, as a function of θ, the case that $\min(N_1,N_2) = 0$ not being excluded. Then

$$\mathcal{C} = \{(x,y) = (\rho(|\theta|)\cos\theta, \rho(|\theta|)\sin\theta) : \ \theta \in [-\pi,\pi]\ \}$$

defines a smooth curve in the (x,y)-plane, symmetric with respect to the x-axis. Rotating $\mathcal{C}$ around the latter, we obtain a smooth surface in (x,y,z)-space, which we call $\partial\mathcal{B}$, where $\mathcal{B}$ is the compact convex body bounded by $\partial\mathcal{B}$. We denote by a_1, a_2 the minimal, resp., maximal x-coordinate on $\partial\mathcal{B}$. Obviously, the Gaussian curvature of $\partial\mathcal{B}$ vanishes at most in the points of intersection with the x-axis.

Theorem. *Suppose that the conditions stated are satisfied, in particular, that the Gaussian curvature of $\partial\mathcal{R}$ vanishes at most in the two points of intersection with the axis of rotation. Then for the number $A_{\mathcal{R}}(t)$ of lattice points in the linearly dilated body $t\mathcal{R}$ the asymptotic formula*

$$A_{\mathcal{R}}(t) = \mathrm{vol}(\mathcal{R})\,t^3 + \sum_{j=2}^{N_1+1} d^*_{1,j}\,\mathcal{F}(-a_1 t, j/(N_1+2))\,t^{2-j/(N_1+2)} -$$
$$- \sum_{j=2}^{N_2+1} d^*_{2,j}\,\mathcal{F}(a_2 t, j/(N_2+2))\,t^{2-j/(N_2+2)} + \Delta_{\mathcal{R}}(t)\,,$$

holds true, where

$$\mathcal{F}(\xi,\eta) := (2\pi)^\eta\,\Gamma(\eta) \sum_{k=1}^{\infty} k^{-1-\eta}\, \sin(2\pi k\xi - \tfrac{1}{2}\pi\eta) \qquad (\eta > 0)\,,$$

and the remainder satisfies the mean-square estimate

$$\int_0^T \left(\Delta_{\mathcal{R}}(t)\right)^2\ \mathrm{d}t = O\left(T^{3+\varepsilon}\right)$$

for each $\varepsilon > 0$. The coefficients $d^*_{1,j}$, $d^*_{2,j}$ are computable, on the basis of the formulas (3.1) - (3.5) *below. In particular, $d^*_{1,1} > 0$, $d^*_{2,1} < 0$.*

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(1) Recall that the curvature of a curve whose equation in polar coordinates is $\rho = \rho(\theta)$, is given, in absolute value, by $\left|\rho(\theta)\,\rho''(\theta) - 2\rho'^2(\theta) - \rho^2(\theta)\right| / (\rho^2(\theta) + \rho'^2(\theta))^{3/2}$.

Remarks. 1. It is easy to see that the error term satisfies in fact the pointwise upper estimate

$$\Delta_{\mathcal{R}}(t) = O\left(t^{3/2+\varepsilon}\right) . \tag{2.3}$$

This is a straightforward consequence of the works of Krätzel [15], [16], [17], [18], but follows also as a simple by-result from the argument in this paper: See the concluding remark at the end.

2. There is a crucial difference in the treatment of the problem, depending on whether there are boundary points of curvature zero or not. For κ nonzero, the analysis leading to the results (1.3), (1.4) is based on the asymptotic expansion of the Fourier transform of the indicator function of the body $\mathcal{B}$, which is due to E. Hlawka [5], [6]. In the case that $\inf \kappa = 0$ the latter is not at our disposal. Thus we have to employ a quite different approach which uses a truncated Hardy's identity (Lemma 1) and a transformation of trigonometric sums.

3. Some auxiliary results.

Lemma 1. *For integers $k \geq 0$, let as usual $r(k)$ denote the number of pairs $(m_1, m_2) \in \mathbb{Z}^2$ with $m_1^2 + m_2^2 = k$. For large real parameters X, Y with $\log Y \ll \log X$, and any $\varepsilon > 0$, it then follows that*

$$P(X) := \sum_{0 \leq k \leq X} r(k) - \pi X =$$

$$= \frac{1}{\pi} X^{1/4} \sum_{1 \leq n \leq Y} \frac{r(n)}{n^{3/4}} \cos(2\pi\sqrt{nX} - 3\pi/4) \; + O\left(X^{1/2+\varepsilon}\, Y^{-1/2}\right) + O\left(Y^{\varepsilon}\right) .$$

Proof. This is contained in formula (1.9) of A. Ivić [9]. $\square$

Lemma 2A. *Let $F \in C^4[A,B]$, $G \in C^2[A,B]$, and suppose that, for positive parameters X, Y, Z, we have $B - A \ll X$ and*

$$F^{(j)} \ll X^{2-j} Y^{-1} \qquad \text{for } j = 2,3,4, \quad F'' \geq c_0 Y^{-1}, \quad G^{(j)} \ll X^{-j} Z \qquad \text{for } j = 0,1,2,$$

throughout the interval $[A,B]$, with some constant $c_0 > 0$. Assume further that there exists a value $\tau_0 \in]A,B[$ with $F'(\tau_0) = 0$. Writing as usual $e(z) := e^{2\pi \mathrm{i} z}$, it follows that

$$\int_A^B G(\tau) e(F(\tau))\, \mathrm{d}\tau = \frac{G(\tau_0)}{\sqrt{F''(\tau_0)}} e(F(\tau_0) + 1/8) + O\left(X^{-1} Y Z\right) +$$

$$+ O\left(Z \,\min\left(|F'(A)|^{-1}, \sqrt{Y}\right)\right) + O\left(Z \,\min\left(|F'(B)|^{-1}, \sqrt{Y}\right)\right) .$$

Proof. This is Lemma III.2 in A.A. Karatsuba and S.M. Voronin [12]. ∎

Lemma 2B. *Let $F \in C^4[A,B]$, $G \in C^2[A,B]$, and suppose that, for positive parameters X, Y, Z, we have $1 \le B - A \ll X$ and*

$$F^{(j)} \ll X^{2-j} Y^{-1} \qquad \text{for } j = 2,3,4, \ |F''| \ge c_0 Y^{-1}, \quad G^{(j)} \ll X^{-j} Z \qquad \text{for } j = 0,1,2,$$

throughout the interval $[A,B]$, with some constant $c_0 > 0$. Let $\mathcal{J}'$ denote the image of $[A,B]$ under F', and F^* the inverse function of F'. Then

$$\sum_{A<m\le B} G(m)\, e(F(m)) = e\left(\frac{\operatorname{sgn}(F'')}{8}\right) \sum_{k\in\mathcal{J}'} \frac{G(F^*(k))}{\sqrt{|F''(F^*(k))|}}\, e\left(F(F^*(k)) - kF^*(k)\right) +$$
$$+ O\left(Z\left(\sqrt{Y} + 1 + \frac{Y}{X} + \sum_{\Lambda=X,Y,Z} |\log\Lambda|\right)\right).$$

Proof. This is essentially contained in Theorem 2.11 of E. Krätzel's monograph [14], apart from his cumbersome condition (2.37) which basically requires the function F to be algebraic. But this can be avoided by replacing, in Krätzel's proof, his Lemma 2.5 by the result we just stated as Lemma 2A. ∎

For our argument it will be essential to have at hand a close analysis of the situation near the points were the Gaussian curvature (possibly) vanishes. To this end, let $\mathcal{C}^+$ denote the upper half of $\mathcal{C}$, and set

$$\mathcal{C}^+ = \{(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta) : \ \theta \in [0,\pi]\ \} = \{(x,y) : \ a_1 \le x \le a_2\,, \ y = f(x)\ \},$$

with $a_1 := -\rho(\pi) < 0 < a_2 := \rho(0)$. This defines $f : \ [a_1, a_2] \to \mathbb{R}_{\ge 0}$ as a strictly positive C^4-function on $]a_1, a_2[$, with $f(a_1) = f(a_2) = 0$, and f'' strictly negative throughout. By our assumptions, for each of the a_i's, and $(x,y) \in \mathcal{C}^+$ in a suitable neighborhood of $(a_i, 0)$,

$$x = a_i + c_i\, y^{N_i+2} + \sum_{m=1}^{\infty} c_{i,m}\, y^{N_i+2+m} \qquad (c_i \ne 0)\,. \tag{3.1}$$

Consequently,

$$\begin{gathered} y = f(x) = \sum_{j=1}^{\infty} d_{i,j}\, |x - a_i|^{\alpha_i\, j}\,, \\ \alpha_i := \frac{1}{N_i+2}\,, \ d_{i,1} = |c_i|^{-\alpha_i} \ne 0\,, \end{gathered} \tag{3.2}$$

and the other $d_{i,j}$'s can be computed recursively from the $c_{i,m}$'s. It thus follows that, for $r = 0, 1, 2, \ldots$,

$$f^{(r)}(x) \asymp |x - a_i|^{\alpha_i - r} \tag{3.3}$$

for x close to a_i. Similarly, we deduce that

$$\frac{\mathrm{d}^r}{\mathrm{d}x^r}\left(\sqrt{f(x)}\right) \asymp |x - a_i|^{\alpha_i/2 - r} \tag{3.4}$$

for $r = 0, 1, 2, \ldots$, and x near a_i. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^2(x)\right) = \sum_{j=2}^{\infty} d^*_{i,j}|x - a_i|^{\alpha_i\, j - 1}\,, \tag{3.5}$$

with $d^*_{i,2} = 2\alpha_i d^2_{i,1}(-1)^{i+1}$, again in an appropriate neighborhood of a_i.

For our proof we will also need some knowledge about the *tac-function* of $\mathcal{R}$

$$H(u, v, w) := \max_{(x,y,z)\in\mathcal{R}} (ux + vy + wz)$$

and the *polar body* $\mathcal{R}^* : \ H(u, v, w) \leq 1$. The connection between the respective smoothness and the curvature of $\partial\mathcal{R}$ and of $\partial\mathcal{R}^*$ has been neatly worked out in W. Müller [23], Lemma 1. It is clear that H and thus $\mathcal{R}^*$ is again invariant under rotations around the first coordinate axis. Let C^*_+ denote the intersection of $\partial\mathcal{R}^*$ with the closed upper half of the (u, v)-plane. Then

$$C^*_+ = \{\ (u, v) :\ 1/a_1 \leq u \leq 1/a_2\,,\ v = h(u)\}\,,$$

where $h : \ [1/a_1, 1/a_2] \to \mathbb{R}_{\geq 0}$ is a strictly positive C^3-function on $]1/a_1, 1/a_2[$, with $h(1/a_1) = h(1/a_2) = 0$ (cf. W. Müller [23], Lemma 1).

Lemma 3A. *With the conditions and definitions stated,*

$$\sup_{1/a_1 < u < 1/a_2} |h(u)\, h'(u)| < \infty\,.$$

Proof. Let the real numbers $x \in\,]a_1, a_2[$ and $u \in\,]1/a_1, 1/a_2[$ be connected by the condition[2]

$$1 = H(u, h(u), 0) = \max_{\xi\in[a_1,a_2]} (u\xi + h(u)f(\xi)) = ux + h(u)f(x)\,.$$

Plainly, $x \to a_i$ if $u \to 1/a_i$, and vice versa. Eliminating $h(u)$ from the pair of equations

$$\begin{aligned} ux + h(u)f(x) &= 1\,, \\ u + h(u)f'(x) &= 0\,, \end{aligned} \tag{3.6}$$

[2] In fact, the points $(x, f(x))$ and $(u, h(u))$ are called *polar reciprocal* to each other. The correspondence defined is one-one.

we get

$$u = \frac{f'(x)}{x\,f'(x) - f(x)}\,. \tag{3.7}$$

By a routine computation,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-f(x)\,f''(x)}{(x\,f'(x)-f(x))^2} \asymp 1 \qquad \text{as } x \to a_i\,,$$

in view of (3.3). By the second part of (3.6),

$$h(u) = -\frac{u}{f'(x)} \asymp |x-a_i|^{1-\alpha_i} \qquad \text{as } x \to a_i\,. \tag{3.8}$$

Therefore, using again (3.7) and (3.3),

$$\begin{aligned} h'(u) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{u}{f'(x)}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^{-1} \asymp \left|\frac{\mathrm{d}}{\mathrm{d}x}\left((-x\,f'(x)+f(x))^{-1}\right)\right| \\ &= \left|\frac{x\,f''(x)}{(x\,f'(x)-f(x))^2}\right| \asymp |x-a_i|^{-\alpha_i} \qquad \text{as } x \to a_i\,. \end{aligned}$$

Together with (3.8) this implies that $h(u)\,h'(u) \asymp |x-a_i|^{1-2\alpha_i}$ as $x \to a_i$, which because of $\alpha_i \le \frac{1}{2}$ proves Lemma 3A. $\square$

Lemma 3B. *For a large real parameter X and the tac-function H defined above, the asymptotics*

$$N(X) := \#\{(m,n) \in \mathbb{Z}\times\mathbb{Z}_{\ge 0}:\ H(m,\sqrt{n},0) \le X\ \} = C\,X^3 + O\,(X) \tag{3.9}$$

holds true, with a certain constant $C > 0$. Furthermore, for $0 < \delta < 1$,

$$\begin{aligned} N_\delta(X) := \#\{(m,n) \in \mathbb{Z}\times\mathbb{Z}_{\ge 0}:\ H(m,\sqrt{n},0) \le X\,,\ \sqrt{n} \le \delta|m|\ \} = \\ = C_\delta\,X^3 + O\,(X)\,, \end{aligned} \tag{3.10}$$

with a positive $C_\delta \ll \delta^2$, the O-constant independent of δ. As a consequence, for large X and $0 < \omega < 1$, $0 < \delta < 1$, it follows that

$$\begin{aligned} &\#\{(m,n) \in \mathbb{Z}\times\mathbb{Z}_{\ge 0}:\ \left|H(m,\sqrt{n},0) - X\right| < \omega\ \} \ll X^2\,\omega + X\,, \\ &\#\{(m,n) \in \mathbb{Z}\times\mathbb{Z}_{\ge 0}:\ \left|H(m,\sqrt{n},0) - X\right| < \omega\,,\ \sqrt{n} \le \delta|m|\,\} \ll X^2\,\omega\,\delta^2 + X\,. \end{aligned} \tag{3.11}$$

Proof. Let $\mathcal{D}_+^*$ denote the compact planar domain bounded by the curve $\mathcal{C}_+^*$ and the u-axis. Obviously,

$$\begin{aligned} N(X) &= \#\{(m,n) \in \mathbb{Z}\times\mathbb{Z}_{\ge 0}:\ (m,\sqrt{n}) \in X\,\mathcal{D}_+^*\ \} = \\ &= \sum_{(1/a_1)X \le m \le (1/a_2)X} \left(1 + [X^2\,h^2(m/X)]\right) = \\ &= X^2 \sum_{(1/a_1)X \le m \le (1/a_2)X} h^2(m/X)\ + O(X) = \\ &= X^2 \int\limits_{(1/a_1)X}^{(1/a_2)X} h^2(u/X)\,\mathrm{d}u + 2X \int\limits_{(1/a_1)X}^{(1/a_2)X} \psi(u) h(u/X) h'(u/X)\,\mathrm{d}u\ + O(X)\,, \end{aligned}$$

by the Euler-Mac Laurin formula (see E. Krätzel [14], p. 20), with $\psi(u) := u - [u] - \frac{1}{2}$. Here the first integral equals $C\,X$ with $C = \int_{1/a_1}^{1/a_2} h^2(\xi)\,\mathrm{d}\xi$, which yields the main term of (3.9). Further, for any interval $[\beta_1, \beta_2] \subset\,]1/a_1, 1/a_2[$, an integration by parts gives

$$\begin{aligned} &2X \int\limits_{\beta_1 X}^{\beta_2 X} \psi(u) h(u/X) h'(u/X)\,\mathrm{d}u = \\ =&2X\left(h(\beta_2)h'(\beta_2)\psi_1(\beta_2 X) - h(\beta_1)h'(\beta_1)\psi_1(\beta_1 X)\right) - \\ &-2\int\limits_{\beta_1 X}^{\beta_2 X} \left(h(u/X)h''(u/X) + h'^2(u/X)\right)\psi_1(u)\,\mathrm{d}u\,, \end{aligned} \tag{3.12}$$

where $\psi_1(u) := \int_0^u \psi(v)\,\mathrm{d}v \ll 1$. If $hh'' + h'^2 = (hh')'$ is bounded on $]1/a_1, 1/a_2[$, we simply let $\beta_1 \to 1/a_1$, $\beta_2 \to 1/a_2$, and obtain the desired bound $O(X)$ for the remainder. In case that $(hh')'$ is unbounded(3) near $1/a_1$ (say), we choose $\beta_1 > 1/a_1$ such that $(hh')'$ has no sign change on $]1/a_1, \beta_1]$. By the second mean-value theorem and Lemma 3A,

$$\int\limits_{(1/a_1)X}^{\beta_1 X} \psi(u) h(u/X) h'(u/X)\,\mathrm{d}u \ll \sup_{]1/a_1, \beta_1]} |hh'| \ll 1\,.$$

A similar reasoning holds near $1/a_2$ if necessary. On the remaining interval $[\beta_1 X, \beta_2 X]$, (3.12) readily yields the bound $O(X)$ and thus completes the proof of (3.9). Quite similarly,

$$\begin{aligned} N_\delta(X) = &\sum_{(1/a_1)X \le m \le (1/a_2)X} \min\left(X^2 h^2(m/X), \delta^2 m^2\right) + O(X) = \\ = &\int\limits_{(1/a_1)X}^{(1/a_2)X} \min\left(X^2 h^2(u/X), \delta^2 u^2\right)\,\mathrm{d}u\ + \\ + &\int\limits_{(1/a_1)X}^{(1/a_2)X} \psi(u)\,\frac{\mathrm{d}}{\mathrm{d}u}\left(\min\left(X^2 h^2(u/X), \delta^2 u^2\right)\right)\,\mathrm{d}u\ + O(X)\,. \end{aligned}$$

Here the first integral obviously equals $C_\delta\, X^3$ with

$$C_\delta := \int\limits_{1/a_1}^{1/a_2} \min\left(h^2(\xi), \delta^2 \xi^2\right)\,\mathrm{d}\xi \ll \delta^2\,.$$

(3) By construction, in particular in view of (3.2), (3.7), (3.8), for u close to $1/a_i$, $(h(u)h'(u))'$ can be represented as a Laurent series in a fractional power of $|x - a_i|$. Thus for $u \to 1/a_i$, $|(h(u)h'(u))'|$ either is bounded or tends to ∞.

The remainder integral can be treated as before, with the bound $O(X)$, since for any interval $I \subseteq]X/a_1, X/a_2[$,

$$\int\limits_I \psi(u)\,\delta^2\,u\,\mathrm{d}u \ll X\,.$$

The deduction of (3.11) from (3.9), (3.10) is trivial. □

4. Asymptotic evaluation of the main terms. For a large parameter t it follows, with the definitions of section 3, that

$$\begin{aligned} A_{\mathcal{R}}(t) &= \sum_{a_1 t \le m \le a_2 t} \left(\sum_{0 \le k \le t^2 f^2(m/t)} r(k) \right) = \\ &= \pi t^2 \sum_{a_1 t \le m \le a_2 t} f^2(m/t) + \sum_{a_1 t \le m \le a_2 t} P(t^2 f^2(m/t))\,. \end{aligned} \tag{4.1}$$

We proceed to evaluate the first sum on the right-hand side, postponing the mean-square estimation of the last one to the next section. By the Euler-Mac Laurin formula,

$$\begin{aligned} \pi t^2 \sum_{a_1 t \le m \le a_2 t} f^2(m/t) &= \pi t^2 \int\limits_{a_1 t}^{a_2 t} f^2(\tau/t)\,\mathrm{d}\tau + \pi t^2 \int\limits_{a_1 t}^{a_2 t} \psi(\tau)\,\frac{\mathrm{d}}{\mathrm{d}\tau}(f^2(\tau/t))\,\mathrm{d}\tau = \\ &= \mathrm{vol}(\mathcal{R}) t^3 + \pi t^2 \int\limits_{a_1}^{a_2} \psi(tx)\,\frac{\mathrm{d}}{\mathrm{d}x}(f^2(x))\,\mathrm{d}x\,. \end{aligned} \tag{4.2}$$

By (3.5),

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^2(x)\right) = \sum_{j=2}^{2N_1+3} d^*_{1,j}(x-a_1)^{\alpha_1 j - 1} + \sum_{j=2}^{2N_2+3} d^*_{2,j}(a_2-x)^{\alpha_2 j - 1} + \Phi(x)\,, \tag{4.3}$$

with $\Phi \in C^1[a_1, a_2]$. Integrating by parts and using again $\psi_1(u) := \int_0^u \psi(v)\,\mathrm{d}v \ll 1$, we obtain

$$\begin{aligned} t^2 \int\limits_{a_1}^{a_2} \psi(tx)\Phi(x)\,\mathrm{d}x &= t\left(\psi_1(a_2 t)\Phi(a_2) - \psi_1(a_1 t)\Phi(a_1)\right) - \\ &- t \int\limits_{a_1}^{a_2} \psi_1(tx)\Phi'(x)\,\mathrm{d}x = O(t)\,. \end{aligned} \tag{4.4}$$

The same argument works for $|x-a_i|^{\alpha_i j - 1}$ instead of $\Phi(x)$, with $N_i + 2 \le j \le 2N_i + 3$, $i \in \{1, 2\}$. In the subsequent analysis we may thus replace the upper summation limits

in the sums from (4.3) by N_1+1, resp., N_2+1. To deal with the first one of these remaining sums, we use the Fourier series

$$\psi(z) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\,\sin(2\pi kz) \qquad (z\notin\mathbb{Z})$$

and an obvious shift of variable. For $j=2,\ldots,N_1+1$, we conclude that

$$\int\limits_{a_1}^{a_2}\psi(tx)(x-a_1)^{\alpha_1 j-1}\,\mathrm{d}x = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\int\limits_{0}^{a_2-a_1} x^{\alpha_1 j-1}\,\sin(2\pi kt(a_1+x))\,\mathrm{d}x\,.$$

An integration by parts shows that

$$\int\limits_{a_2-a_1}^{\infty} x^{\alpha_1 j-1}\,\sin(2\pi kt(a_1+x))\,\mathrm{d}x = O\left((kt)^{-1}\right)\,.$$

Further,

$$\int\limits_{0}^{\infty} x^{\alpha_1 j-1}\,\sin(2\pi kt(a_1+x))\,\mathrm{d}x = \Im\left(e(a_1kt)\,(kt)^{-\alpha_1 j}\int\limits_{0}^{\infty}\tau^{\alpha_1 j-1}e(\tau)\,\mathrm{d}\tau\right) =$$

$$= \Im\left(e(a_1kt)\,(2\pi kt)^{-\alpha_1 j}\Gamma(\alpha_1 j)\,e(\alpha_1 j/4)\right) = \Gamma(\alpha_1 j)\,(2\pi kt)^{-\alpha_1 j}\,\sin\left(2\pi a_1kt+\tfrac{1}{2}\pi\alpha_1 j\right)\,,$$

using well-known formulas for the last integral (cf., e.g., H. Rademacher [28], p. 82). Collecting results, we get

$$\begin{aligned}
&\pi t^2\int\limits_{a_1}^{a_2}\psi(tx)\left(\sum_{j=2}^{N_1+1} d^*_{1,j}(x-a_1)^{\alpha_1 j-1}\right)\mathrm{d}x = \\
&= \sum_{j=2}^{N_1+1} d^*_{1,j}\,t^{2-\alpha_1 j}(2\pi)^{-\alpha_1 j}\,\Gamma(\alpha_1 j)\sum_{k=1}^{\infty}k^{-1-\alpha_1 j}\,\sin(-2\pi a_1kt-\tfrac{1}{2}\pi\alpha_1 j)+O\left(t\right) = \\
&= \sum_{j=2}^{N_1+1} d^*_{1,j}\,\mathcal{F}(-a_1t,\alpha_1 j)\,t^{2-\alpha_1 j}+O\left(t\right)\,,
\end{aligned}$$

where $\mathcal{F}(\cdot,\cdot)$ has been defined in our Theorem. Quite similarly,

$$\begin{aligned}
&\pi t^2\int\limits_{a_1}^{a_2}\psi(tx)\left(\sum_{j=2}^{N_2+1} d^*_{2,j}(a_2-x)^{\alpha_2 j-1}\right)\mathrm{d}x = \\
&= -\sum_{j=2}^{N_2+1} d^*_{2,j}\,\mathcal{F}(a_2t,\alpha_2 j)\,t^{2-\alpha_2 j}+O\left(t\right)\,.
\end{aligned}$$

Combining the last two results with (4.2) - (4.4), we finally arrive at

$$\begin{aligned} \pi t^2 \sum_{a_1 t\le m\le a_2 t} f^2(m/t) = \mathrm{vol}(\mathcal{R})\, t^3 + \sum_{j=2}^{N_1+1} d^*_{1,j}\, \mathcal{F}(-a_1 t, \alpha_1 j)\, t^{2-\alpha_1 j} - \\ - \sum_{j=2}^{N_2+1} d^*_{2,j}\, \mathcal{F}(a_2 t, \alpha_2 j)\, t^{2-\alpha_2 j} + O\left(t\right) . \end{aligned} \tag{4.5}$$

5. Estimating the remainder in mean-square. It remains to deal with the last sum in (4.1), i.e., to show that

$$\int_T^{2T} \left(\sum_{a_1 t\le m\le a_2 t} P\left(t^2\, f^2(m/t)\right)\right)^2 \,\mathrm{d}t \ll T^{3+\varepsilon} .$$

For given large T we divide the intervals $]a_1, \frac{1}{2}(a_1+a_2)]$ and $]\frac{1}{2}(a_1+a_2), a_2]$ into dyadic subintervals $\mathcal{J}^{(1,r)} =]u^{(1,r+1)}, u^{(1,r)}]$, $\mathcal{J}^{(2,r)} =]u^{(2,r)}, u^{(2,r+1)}]$, $0\le r\le R$, where $u^{(i,r)} := a_i - (-1)^i\, 2^{-r-1}(a_2-a_1)$, and R is chosen such that the shortest ones of these intervals are of length $\asymp T^{-1}$. Ignoring the superscripts for short, we write $\mathcal{J}$ for any of these subintervals, whose number obviously is $O(\log T)$. Let $\mathcal{K} := [a_1, a_2]\setminus \bigcup \mathcal{J}$, then $|\mathcal{K}| \asymp T^{-1}$, and the trivial bound $P(z)\ll \sqrt{z}$ readily implies

$$\int_T^{2T} \left(\sum_{m\in t\mathcal{K}} P\left(t^2\, f^2(m/t)\right)\right)^2 \,\mathrm{d}t \ll T^{3} .$$

Thus it suffices to prove that, for each $\mathcal{J}$ and $\varepsilon>0$,

$$\int_T^{2T} \left(\sum_{m\in t\mathcal{J}} P\left(t^2\, f^2(m/t)\right)\right)^2 \,\mathrm{d}t \ll T^{3+\varepsilon} . \tag{5.1}$$

For every $t\in[T,2T]$ and $m\in t\mathcal{J}$, we apply Lemma 1, with $X = t^2\, f^2(m/t)$ and $Y = T^2$. By (3.3), $X \gg T^2 |\mathcal{K}|^{2\max(\alpha_i)} \gg T$, hence the condition $\log Y \ll \log X$ is satisfied. We obtain

$$\begin{aligned} &\sum_{m\in t\mathcal{J}} P\left(t^2\, f^2(m/t)\right) = \\ &= \sum_{m\in t\mathcal{J}} \left(\frac{\sqrt{t}}{\pi} \sqrt{f\left(\frac{m}{t}\right)} \sum_{1\le n\le T^2} \frac{r(n)}{n^{3/4}} \cos(2\pi\sqrt{n}\, t f(m/t) - 3\pi/4) + O\left(T^{\varepsilon}\right)\right) = \\ &= \frac{\sqrt{t}}{\pi} \sum_{1\le n\le T^2} \frac{r(n)}{n^{3/4}} \left\{ \sum_{m\in t\mathcal{J}} \sqrt{f\left(\frac{m}{t}\right)} \cos(2\pi\sqrt{n}\, t f(m/t) - 3\pi/4)\right\} + O\left(T^{1+\varepsilon}\right) . \end{aligned} \tag{5.2}$$

We shall transform the inner sum here by means of Lemma 2B, with

$$G(\tau) := \sqrt{f\left(\tau/t\right)}\,, \quad F(\tau) := \sqrt{n}\, t f(\tau/t)\,.$$

To do so we put $L := T\,\mathrm{length}(\mathcal{J})$, and observe that $|\tau - at| \asymp L$ for all $\tau \in t\mathcal{J}$, where a is the one of a_1, a_2 which is nearer to $\mathcal{J}$. Hence, in view of (3.3),

$$F''(\tau) = \sqrt{n}\, t^{-1} f''(\tau/t) \asymp \sqrt{n}\, t^{-1} |\tau/t - a|^{\alpha-2} \asymp \sqrt{n}\, t^{1-\alpha} L^{\alpha-2}$$

for all $\tau \in t\mathcal{J}$ (α the appropriate one of α_1, α_2), and similarly

$$F^{(j)}(\tau) \ll \sqrt{n}\, t^{1-\alpha} L^{\alpha-j} \qquad \text{for } j = 3, 4\,.$$

Furthermore, by (3.4),

$$G^{(j)}(\tau)) = \frac{\mathrm{d}^j}{\mathrm{d}\tau^j}\left(\sqrt{f(\tau/t)}\right) \ll t^{-j}\, |\tau/t - a|^{\alpha/2-j} \ll t^{-\alpha/2}\, L^{\alpha/2-j}$$

for $\tau \in t\mathcal{J}$, $j = 0, 1, 2$. We may thus apply Lemma 2B with the parameters

$$X := L\,, \quad Y := n^{-1/2}\, t^{\alpha-1}\, L^{2-\alpha}\,, \quad Z := (L/t)^{\alpha/2}\,.$$

After a short computation, Lemma 2B yields

$$\begin{aligned} &\sum_{m \in t\mathcal{J}} \sqrt{f\left(\frac{m}{t}\right)} \cos(2\pi\sqrt{n}\, t f(m/t) - 3\pi/4) = \\ &= \frac{\sqrt{t}}{n^{1/4}} \Re \left(e(-\tfrac12) \sum_{k \in \sqrt{n}\mathcal{J}^*} \frac{\sqrt{f(f^*(k/\sqrt{n}))}}{\left|f''(f^*(k/\sqrt{n}))\right|^{1/2}}\, e\left(t(\sqrt{n} f(f^*(k/\sqrt{n})) - k f^*(k/\sqrt{n}))\right) \right) + \\ &\quad + \; O\left(n^{-1/4}\, t^{(\alpha-1)/2}\, L^{1-\alpha/2}\right) + O\left(\log t\right)\,, \end{aligned} \tag{5.3}$$

where f^* denotes the inverse function of f' and $\mathcal{J}^*$ the image of the closure $\bar{\mathcal{J}}$ of $\mathcal{J}$ under f'. The contribution of the error terms here to the whole of (5.2) is

$$\ll t \sum_{1 \le n \le T^2} \frac{r(n)}{n} + \sqrt{t}\, \log t \sum_{1 \le n \le T^2} \frac{r(n)}{n^{3/4}} \ll T\, \log T\,,$$

hence small enough. We put for short $\beta(k,n) := \frac{\sqrt{f(f^*(k/\sqrt{n}))}}{\left|f''(f^*(k/\sqrt{n}))\right|^{1/2}}$. Now $k \in \sqrt{n}\mathcal{J}^*$ implies that $f^*(k/\sqrt{n}) \in \bar{\mathcal{J}}$. Hence, by (3.3) and the fact that f'' is bounded away from zero,

$$\beta(k,n) \ll \frac{L}{T} \qquad \text{for } k \in \sqrt{n}\mathcal{J}^*\,. \tag{5.4}$$

Furthermore, by the definition of the tac-function H, for all $k \in \mathbb{Z}, n \in \mathbb{Z}^+$,

$$H(-k, \sqrt{n}, 0) = \max_{(x,y,0)\in\partial\mathcal{R}} \left(-kx + \sqrt{n}\, y\right) = \max_{a_1 \le x \le a_2} \left(-kx + \sqrt{n} f(x)\right) = \\ = \sqrt{n} f(f^*(k/\sqrt{n})) - k f^*(k/\sqrt{n})\,.$$

Hence it will suffice to show that

$$I(\mathcal{J}, T) := \int\limits_T^{2T} t^2 \left|S(\mathcal{J}, t, T)\right|^2 \,\mathrm{d}t \ll T^{3+\varepsilon}\,, \tag{5.5}$$

where

$$S(\mathcal{J}, t, T) := \sum_{1 \le n \le T^2} \frac{r(n)}{n} \sum_{k \in \sqrt{n}\,\mathcal{J}^*} \beta(k,n)\, e(t\, H(-k, \sqrt{n}, 0))\,.$$

To simplify the subsequent analysis, we use a common device involving the Fejér kernel $\phi(z) := \sin^2(\pi z)/(\pi z)^2$. By Jordan's inequality, $\phi(z) \ge 4/\pi^2$ for $|z| \le \frac{1}{2}$, and the Fourier transform is simply

$$\widehat{\phi}(y) = \int\limits_{\mathbb{R}} \phi(z)\, e(yz)\,\mathrm{d}y = \max(0, 1 - |y|)\,. \tag{5.6}$$

Thus

$$I(\mathcal{J}, T) \le 4T^3 \int\limits_{-1/2}^{1/2} \left|S(\mathcal{J}, 3T/2 + Tw, T)\right|^2 \,\mathrm{d}w \le$$

$$\le \pi^2 T^3 \int\limits_{\mathbb{R}} \phi(w)\, \left|S(\mathcal{J}, 3T/2 + Tw, T)\right|^2 \,\mathrm{d}w =$$

$$= \pi^2 T^3 \sum_{1 \le n, n' \le T^2} \frac{r(n) r(n')}{n n'} \sum_{\substack{k \in \sqrt{n}\,\mathcal{J}^* \\ k' \in \sqrt{n'}\,\mathcal{J}^*}} \beta(k,n)\beta(k',n') \times$$

$$\times e\left(\frac{3T}{2}\,(H(\mathbf{m}) - H(\mathbf{m}'))\right) \widehat{\phi}\left(T(H(\mathbf{m}) - H(\mathbf{m}'))\right)\,,$$

where $\mathbf{m} := (-k, \sqrt{n}, 0)$, $\mathbf{m}' := (-k', \sqrt{n'}, 0)$ for short. Therefore, by (5.4) and (5.6),

$$I(\mathcal{J}, T) \ll\; T^3\,(L/T)^2 \sum_{\substack{1 \le n, n' \le T^2 \\ k/\sqrt{n},\, k'/\sqrt{n'} \in \mathcal{J}^*}} \frac{r(n) r(n')}{n n'} \max\left(0, 1 - T|H(\mathbf{m}) - H(\mathbf{m}')|\right)\,.$$

We observe that

$$\max_{\xi \in \mathcal{J}^*} |\xi| = \max_{x \in \bar{\mathcal{J}}} |f'(x)| \ll (L/T)^{\alpha - 1}\,, \tag{5.7}$$

again by (3.3). We put $\lambda := L/T = \mathrm{length}(\mathcal{J})$ for short. If λ is sufficiently small (i.e., $\mathcal{J}$ is close to an endpoint a_i), all numbers of $\mathcal{J}^*$ are $\asymp \lambda^{\alpha-1}$. Hence $k/\sqrt{n}\in\mathcal{J}^*$ implies that

$$k \asymp \sqrt{n}\,\lambda^{\alpha-1} \quad \text{if } \lambda \text{ is sufficiently small,}$$
$$k \ll \sqrt{n}\,\lambda^{\alpha-1} \quad \text{always.}$$

Furthermore, in any case, for $n\ge 1$ and $k/\sqrt{n}\in\mathcal{J}^*$,

$$H(\mathbf{m}) \asymp |\mathbf{m}|_\infty \asymp \sqrt{n}\,\lambda^{\alpha-1}\,.$$

Thus

$$I(\mathcal{J},T) \ll T^{3+\varepsilon/2}\lambda^{4\alpha-2} \sum_{\substack{1\le n\le T^2\\ k/\sqrt{n}\,\in\mathcal{J}^*}} |\mathbf{m}|_\infty^{-4} \times$$
$$\times\ \#\{\mathbf{m}' :\ |H(\mathbf{m}')-H(\mathbf{m})| < T^{-1}\,,\ k'/\sqrt{n'}\ \in \mathcal{J}^*\ \}\,.$$

By (3.11) of Lemma 3B,

$$\#\{\mathbf{m}' = (-k',\sqrt{n'},0):\ |H(\mathbf{m}')-H(\mathbf{m})| < T^{-1},\ \sqrt{n'} \gg |k'|\lambda^{1-\alpha}\ \} \ll$$
$$\ll |\mathbf{m}|_\infty^2\,T^{-1}\,\lambda^{2(1-\alpha)} + |\mathbf{m}|_\infty\,.$$

Therefore,

$$I(\mathcal{J},T) \ll T^{3+\varepsilon/2} \sum_{\substack{1\le n\le T^2\\ k\ll\sqrt{n}\,\lambda^{\alpha-1}}} \left(\frac{\lambda^{2\alpha}}{|\mathbf{m}|_\infty^2\,T} + \frac{\lambda^{4\alpha-2}}{|\mathbf{m}|_\infty^3}\right) \ll$$
$$\ll T^{3+\varepsilon/2} \sum_{\substack{1\le n\le T^2\\ k\ll\sqrt{n}\,\lambda^{\alpha-1}}} \left(\frac{\lambda^{2}}{n\,T} + \frac{\lambda^{1+\alpha}}{n^{3/2}}\right) \ll$$
$$\ll T^{3+\varepsilon/2} \sum_{1\le n\le T^2} \left(\frac{\lambda^{1+\alpha}}{T\sqrt{n}} + \frac{\lambda^{2\alpha}}{n}\right) \ll T^{3+\varepsilon}\,.$$

This completes the proof of (5.5) and thereby, in view of (4.1) and (4.5), that of our Theorem.

6. Concluding remark. We indicate briefly how the pointwise bound (2.3) follows as a by-result from the above analysis. Estimating the right-hand side of (5.3) trivially, we get, for $t=T$,

$$\sum_{m\in t\mathcal{J}} \sqrt{f\left(\frac{m}{t}\right)}\,\cos(2\pi\sqrt{n}\,t f(m/t) - 3\pi/4) \ll$$
$$\ll n^{1/4}\,t^{1/2}\,|\mathcal{J}^*| + n^{-1/4}\,t^{(\alpha-1)/2}\,L^{1-\alpha/2} + \log t \ll$$
$$\ll n^{1/4}\,t^{3/2-\alpha}\,L^{\alpha-1} + n^{-1/4}\,t^{(\alpha-1)/2}\,L^{1-\alpha/2} + \log t\,,$$

in view of (5.7). Using Lemma 1 with $X = t^2 f^2(m/t) \ll t^2$, $Y = L^{2-\alpha} t^{\alpha-1} \ll t$, we obtain as an obvious variant of (5.2)

$$\sum_{m \in t\mathcal{J}} P\left(t^2 f^2(m/t)\right) \ll$$
$$\ll \sqrt{t} \sum_{1 \le n \le Y} \frac{r(n)}{n^{3/4}} \left(n^{1/4} t^{3/2-\alpha} L^{\alpha-1} + n^{-1/4} t^{(\alpha-1)/2} L^{1-\alpha/2} + \log t\right) + \frac{L\, t^{1+\varepsilon}}{Y^{1/2}} \ll$$
$$\ll t^{2-\alpha} L^{\alpha-1} Y^{1/2} + t^{\alpha/2} L^{1-\alpha/2} \log t + t^{1/2} Y^{1/4} \log t + \frac{L\, t^{1+\varepsilon}}{Y^{1/2}} \ll$$
$$\ll L^{\alpha/2} t^{3/2-\alpha/2+\varepsilon} + L^{1-\alpha/2} t^{\alpha/2} \log t + L^{1/2-\alpha/4} t^{\alpha/4+1/4} \log t \ll t^{3/2+\varepsilon}. \quad \square$$

References

[1] V. Bentkus and F. Götze, On the lattice point problem for ellipsoids. Acta Arithm. **80**, 101-125 (1997).

[2] F. Chamizo, Lattice points in bodies of revolution. Acta Arithm. **85**, 265-277 (1998).

[3] F. Götze, Lattice point problems and values of quadratic forms. Invent. Math. **157**, 195–226 (2004).

[4] K. Haberland, Über die Anzahl der Gitterpunkte in konvexen Gebieten. Preprint FSU Jena 1993 (unpublished).

[5] E. Hlawka, Über Integrale auf konvexen Körpern I. Monatsh. f. Math. **54**, 1-36 (1950).

[6] E. Hlawka, Über Integrale auf konvexen Körpern II. Monatsh. f. Math. **54**, 81–99 (1950).

[7] M.N. Huxley, Area, lattice points, and exponential sums. LMS Monographs, New Ser. **13**, Oxford 1996.

[8] A. Iosevich, E. Sawyer, and A. Seeger, Mean square discrepancy bounds for the number of lattice points in large convex bodies. J. Anal. Math. **87**, 209-230 (2002).

[9] A. Ivić, The Laplace transform of the square in the circle and divisor problems, Stud. Sci. Math. Hung. **32**, 181-205 (1996).

[10] A. Ivić, E. Krätzel, M. Kühleitner, and W.G. Nowak, Lattice points in large regions and related arithmetic functions: Recent developments in a very classic topic. Proceedings Conf. on Elementary and Analytic Number Theory ELAZ'04, held in Mainz, May 24-28, W. Schwarz and J. Steuding eds., Franz Steiner Verlag 2006, pp. 89-128.

[11] V. Jarnik, Über die Mittelwertsätze der Gitterpunktlehre, V. Abh. Cas. mat. fys. **69**, 148–174 (1940).

[12] A.A. Karatsuba and S.M. Voronin, The Riemann zeta-function. Berlin 1992.

[13] D.G. Kendall, On the number of lattice points inside a random oval. Quart. J. Math. (Oxford) **19**, 1-26 (1948).

[14] E. Krätzel, Lattice points. Berlin 1988.

[15] E. Krätzel, Analytische Funktionen in der Zahlentheorie. Stuttgart-Leipzig-Wiesbaden 2000.

[16] E. Krätzel, Lattice points in three-dimensional large convex bodies. Math. Nachr. **212**, 77–90 (2000).

[17] E. Krätzel, Lattice points in three-dimensional convex bodies with points of Gaussian curvature zero at the boundary. Monatsh. Math. **137**, 197–211 (2002).

[18] E. Krätzel, Lattice points in some special three-dimensional convex bodies with points of Gaussian curvature zero at the boundary. Comment. Math. Univ. Carolinae **43**, 755-771 (2002).

[19] E. Krätzel and W.G. Nowak, Eine explizite Abschätzung für die Gitter-Diskrepanz von Rotationsellipsoiden. Monatsh. Math., to appear.

[20] M. Kühleitner, Lattice points in bodies of revolution in $\mathbb{R}^3$: an Ω_--estimate for the error term. Arch. Math. (Basel) **74**, 234-240 (2000).

[21] M. Kühleitner and W.G. Nowak, The lattice point discrepancy of a body of revolution: Improving the lower bound by Soundararajan's method. Arch. Math. (Basel) **83**, 208–216 (2004).

[22] Y.-K. Lau, On the mean square formula of the error term for a class of arithmetical functions. Monatsh. Math. **128**, 111-129 (1999).

[23] W. Müller, On the average order of the lattice rest of a convex body. Acta Arithm. **80**, 89–100 (1997).

[24] W. Müller, Lattice points in large convex bodies. Monatsh. Math. **128**, 315–330 (1999).

[25] W.G. Nowak, On the average order of the lattice rest of a convex planar domain. Proc. Cambridge Phil. Soc. **98**, 1–4 (1985).

[26] W.G. Nowak, On the mean lattice point discrepancy of a convex disc. Arch. Math. (Basel) **78**, 241–248 (2002).

[27] M. Peter, Lattice points in convex bodies with planar points on the boundary. Monatsh. Math. **135**, 37–57 (2002).

[28] H. Rademacher, Topics in analytic number theory. Berlin-Heidelberg-New York 1973.

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